

SPECTRAL PROPERTY OF CANTOR MEASURES WITH CONSECUTIVE DIGITS

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ABSTRACT. We consider equally-weighted Cantor measures $\mu_{q,b}$ arising from iterated function systems of the form $b^{-1}(x+i)$, $i = 0, 1, \dots, q-1$, where $q < b$. We classify the (q, b) so that they have infinitely many mutually orthogonal exponentials in $L^2(\mu_{q,b})$. In particular, if q divides b , the measures have a complete orthogonal exponential system and hence spectral measures. Improving the construction in [DHS], we characterize all the maximal orthogonal sets Λ when q divides b via a maximal mapping on the q -adic tree in which all elements in Λ are represented uniquely in finite b -adic expansions and we can separate the maximal orthogonal sets into two types: regular and irregular sets. For a regular maximal orthogonal set, we show that its completeness in $L^2(\mu_{q,b})$ is crucially determined by the certain growth rate of non-zero digits in the tail of the b -adic expansions of the elements. Furthermore, we exhibit complete orthogonal exponentials with zero Beurling dimensions. These examples show that the technical condition in Theorem 3.5 of [DHSW] cannot be removed. For an irregular maximal orthogonal set, we show that under some condition, its completeness is equivalent to that of the corresponding regularized mapping.

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1. INTRODUCTION

Let μ be a compactly supported Borel probability measure on \mathbb{R}^d . We say that μ is a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ so that $E(\Lambda) :=$

1991 *Mathematics Subject Classification.* Primary 28A80; Secondary 42C15.

Key words and phrases. Cantor measures; spectral measures; spectra; trees.

The research is supported in part by the HKRGC Grant and the Focused Investment Scheme of CUHK; The second author is also supported by the National Natural Science Foundation of China 11271148.

$\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a *spectrum* of μ . If $\chi_\Omega dx$ is a spectral measure, then we say that Ω is a *spectral set*. The study of spectral measures was first initiated from B. Fuglede in 1974 [Fu], when he considered a functional analytic problem of extending some commuting partial differential operators to some dense subspace of L^2 functions. In his first attempt, Fuglede proved that any fundamental domains given by a discrete lattice are spectral sets with its dual lattice as its spectrum. On the other hand, he also proved that triangles and circles on \mathbb{R}^2 are not spectral sets, while some examples (e.g. $[0, 1] \cup [2, 3]$) that are not fundamental domains can still be spectral. From the examples and the relation between Fourier series and translation operators, he proposed a reasonable conjecture on spectral sets: *$\Omega \subset \mathbb{R}^s$ is a spectral set if and only if Ω is a translational tile*. This conjecture baffled experts for 30 years until 2004, Tao [T] gave the first counterexamples on \mathbb{R}^d , $d \geq 5$. The examples were modified later so that the conjecture are false in both directions on \mathbb{R}^d , $d \geq 3$ [KM1], [KM2]. It remains open in dimension 1 and 2. Despite the counterexamples, the exact relationship between spectral measures and tiling is still mysterious.

The problem of spectral measures is as exciting when we consider fractal measures. Jorgensen and Pedersen [JP] showed that the standard Cantor measures are spectral measures if the contraction is $\frac{1}{2n}$, while there are at most two orthogonal exponentials when the contraction is $\frac{1}{2n+1}$. Following this discovery, more spectral self-similar/self-affine measures were also found ([LaW], [DJ] et. al.). The construction of these spectral self-similar measures is based on the existence of the *compatible pairs (known also as Hadamard triples)*. It is still unknown whether all such spectral measures are obtained from compatible pairs. Having an exponential basis, the series convergence problem was also studied by Strichartz. It is surprising that the ordinary Fourier series of continuous functions converge uniformly for standard Cantor measures [Str]. By now there are considerable amount of literatures studying spectral measures and other generalized types of Fourier expansions like the Fourier frames and Riesz bases ([DHJ], [DHSW], [DL], [HLL], [IP], [JKS] [La], [LaW], [Lai], [Li1], [Li2], and the references therein).

In [HuL], Hu and Lau made a start in studying the spectral properties of Bernoulli convolutions, the simplest class of self-similar measures. They classified the contraction ratios with infinitely many orthogonal exponentials. It was recently shown by Dai that the only spectral Bernoulli convolutions are of contraction ratio $\frac{1}{2n}$ [D]. In this paper, we study another general class of Cantor measures on \mathbb{R}^1 . Let $b > 2$ be an integer and $q < b$ be another positive integer. We consider the iterated function system(IFS) with maps

$$f_i(x) = b^{-1}(x + i), \quad i = 0, 1, \dots, q-1.$$

The IFS arises a natural *self-similar measure* $\mu = \mu_{q,b}$ satisfying

$$\mu(E) = \sum_{i=0}^{q-1} \frac{1}{q} \mu(f_i^{-1}(E)) \quad (1.1)$$

for all Borel sets E . Note that we only need to consider equal weight since non-equally weighted self-similar measures here cannot have any spectrum by Theorem 1.5 in [DL]. It is also clear that if $q = 2$, μ becomes the standard Cantor measure of b^{-1} contraction. For this class of self-similar measures, we find surprisingly that the spectral properties depend heavily on the number theoretic relationship between q and b . Our first result is to show that $\mu = \mu_{q,b}$ *has infinitely many orthogonal exponentials if and only if q and b is not relatively prime. If moreover, q divides b , the resulting measure will be a spectral measure* (Theorem 2.1). However, when q does not divide b and they are not relatively prime (e.g. $q = 4$, $b = 6$), variety of cases may occur and we are not sure whether there are spectral measures in these classes (see Remark 3.1).

We then focus on the case when $b = qr$ in which we aim at giving a detailed classification of its spectra. The classification of spectra, for a given spectral measure, was first studied by Lagarias, Reeds and Wang [LRW]. They considered the spectra of $L^2([0, 1]^d)$ (more generally fundamental domains of some lattices) and they showed that the spectra of $L^2([0, 1]^d)$ are exactly all the tiling sets of $[0, 1]^d$. If $d = 1$, the way of tiling $[0, 1]$ is rather rigid, and it is easy to see that the only spectrum (respectively the tiling set) is the translates of the integer lattice \mathbb{Z} .

Such kind of rigidity breaks down even on \mathbb{R}^1 if we turn to fractal measures. The first attempt of the classification of its spectra was due to [DHS], Dutkay, Han and Sun decomposed the maximal orthogonal sets of one-fourth Cantor measure using 4-adic expansion with digits $\{0, 1, 2, 3\}$ and put them into a labeling of the binary tree. The maximal orthogonal sets will then be obtained by reading all the infinite paths with digits ending eventually in 0 (for positive elements) or 3 (for negative elements). They also gave some sufficient conditions on the digits for a maximal orthogonal set to be a spectrum. Nonetheless, the condition is not easy to verify.

Turning to our self-similar measures with consecutive digits where the one-fourth Cantor measure is a special case, we will classify all the maximal orthogonal sets using mappings on the standard q -adic tree called *maximal mappings* (Theorem 2.6). This construction improves the tree labeling method in [DHS] in two ways.

- (1) We will choose the digit system to be $\{-1, 0, 1, \dots, b-2\}$ instead of $\{0, 1, \dots, b-1\}$. By doing so, all integers (both positive and negative) can be expanded into b -adic expansions terminating at 0 (Lemma 2.4).
- (2) We impose restrictions on our labeling position on the tree so that together with (1), all the elements in a maximal orthogonal set can be extracted

by reading some specific paths in the tree. These paths are collected in a countable set Γ_q defined in (2.4).

Having such a new tree structure of a maximal orthogonal set, we discover there are two possibilities for the maximal sets depending on whether all the paths in Γ_q are corresponding to some elements in the maximal orthogonal sets (i.e. the values assigned are eventually 0). If it happens that all the paths in Γ_q behave nicely as said, we call such maximal orthogonal sets *regular*. It turns out that regular sets cover most of the interesting cases and we can give regular sets a natural ordering $\{\lambda_n : n = 0, 1, 2, \dots\}$. If the standard q -adic expansion of n has length k , we define N_n^* to be the number of non-zero digits in the b -adic expansion using $\{-1, 0, \dots, b-2\}$ of λ_n after k . N_n^* is our crucial factor in determining whether the set is a spectrum. We show that *if N_n^* grows slowly enough or even uniformly bounded, the set will be a spectrum, while if N_n^* grows too fast, say it is of polynomial rates, then the maximal orthogonal sets will not be a spectrum* (Theorem 2.7).

In [DHSW], Dutkay *et al* tried to generalize the classical results of Landau [Lan] about the Beurling density on Fourier frames to fractal settings. They defined the concept of *Beurling dimensions* for a discrete set and showed that all Bessel sequences for an IFS of similitudes with no overlap condition must have Beurling dimension not greater than its Hausdorff dimension of the attractor. Under technical assumption on the frame spectra, they showed the above two dimensions coincide. They conjectured that the assumption can be removed. However, as we see that N_n^* counts the number of non-zero digits only, we can freely add qb^m for any $m > 0$ on the tree of the canonical spectrum. These additional terms push the λ_n 's as far away from each other as wanted and we therefore show that *there exists spectrum of zero Beurling dimension* (Theorem 2.9).

For the organization of the paper, we present our set-up and main results in Section 2. In Section 3, we discuss the maximal orthogonal sets of $\mu_{q,b}$ and classify all maximal orthogonal sets via the maximal mapping on the q -adic when q divides b . In Section 4, we discuss the regular spectra and prove the growth rate criteria. Moreover, the examples of the spectra with zero Beurling dimensions will be given. In Section 5, we give a study on the irregular spectra.

2. SETUP AND MAIN RESULTS

Let Λ be a countable set in \mathbb{R} and denote $E(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$ where $e_\lambda(x) = e^{2\pi i \lambda x}$. We say that Λ is a *maximal orthogonal set (spectrum)* if $E(\Lambda)$ is a maximal orthogonal set (an orthonormal basis) for $L^2(\mu)$. Here $E(\Lambda)$ is a maximal orthogonal set of exponentials means that it is a mutually orthogonal set in $L^2(\mu)$ such that if $\alpha \notin \Lambda$, e_α is not orthogonal to some e_λ , $\lambda \in \Lambda$. If $L^2(\mu)$ admits a spectrum, then μ

is called a *spectral measure*. Given a measure μ , the Fourier transform is defined to be

$$\widehat{\mu}(\xi) = \int e^{2\pi i \xi x} d\mu(x).$$

It is easy to see that $E(\Lambda)$ is an orthogonal set if and only if

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}) := \{\xi \in \mathbb{R} : \widehat{\mu}(\xi) = 0\}.$$

We call such Λ a *bi-zero set* of μ . For $\mu = \mu_{q,b}$, we can calculate its Fourier transform.

$$\widehat{\mu}(\xi) = \prod_{j=1}^{\infty} \left[\frac{1}{q} (1 + e^{2\pi i b^{-j} \xi} + \dots + e^{2\pi i b^{-j} (q-1) \xi}) \right]. \quad (2.1)$$

Denote

$$m(\xi) = \frac{1}{q} (1 + e^{2\pi i \xi} + \dots + e^{2\pi i (q-1) \xi}) \quad (2.2)$$

and thus $|m(\xi)| = |\frac{\sin q\pi\xi}{q \sin \pi\xi}|$. The zero set of m is

$$\mathcal{Z}(m) = \left\{ \frac{a}{q} : q \nmid a, a \in \mathbb{Z} \right\},$$

where $q \nmid a$ means q does not divide a . We can then write $\widehat{\mu}(\xi) = \prod_{j=1}^{\infty} m(b^{-j}\xi)$, so that the zero set of $\widehat{\mu}$ is given by

$$\mathcal{Z}(\widehat{\mu}) = \left\{ \frac{b^n}{q} a : n \geq 1, q \nmid a \right\} = r \{b^n a : n \geq 0, q \nmid a\}, \quad (2.3)$$

where $r = b/q$.

We have the following theorem classifying which $\mu_{q,b}$ possess infinitely many orthogonal exponentials. It is also the starting point of our paper.

Theorem 2.1. *$\mu = \mu_{q,b}$ has infinitely many orthogonal exponentials if and only if the greatest common divisor between q and b is greater than 1. If q divides b , then $\mu_{q,b}$ is a spectral measure.*

We wish to give a classification on the spectra and the maximal orthogonal sets whenever they exist. To do this, it is convenient to introduce some multiindex notations: Denote $\Sigma_q = \{0, \dots, q-1\}$, $\Sigma_q^0 = \{\vartheta\}$ and $\Sigma_q^n = \underbrace{\Sigma_q \times \dots \times \Sigma_q}_n$. Let

$\Sigma_q^* = \bigcup_{n=0}^{\infty} \Sigma_q^n$ be the set of all finite words and let $\Sigma_q^\infty = \Sigma_q \times \Sigma_q \times \dots$ be the set of all infinite words. Given $\sigma = \sigma_1 \sigma_2 \dots \in \Sigma_q^\infty \cup \Sigma_q^*$, we define $\vartheta \sigma = \sigma$, $\sigma|_k = \sigma_1 \dots \sigma_k$ for $k \geq 0$ where $\sigma|_0 = \vartheta$ for any σ and adopt the notation $0^\infty = 000\dots$, $0^k = \underbrace{0\dots 0}_k$

and $\sigma\sigma'$ is the concatenation of σ and σ' . We start with a definition.

Definition 2.2. Let Σ_q^* be all the finite words defined as above. We say it is a q -adic tree if we set naturally the root is ϑ , all the k -th level nodes are Σ_q^k for $k \geq 1$ and all the offsprings of $\sigma \in \Sigma_q^*$ are σi for $i = 0, 1, \dots, q-1$.

Let τ be a map from Σ_q^* to real numbers. Then the image of τ defines a q -adic tree labeling. Define Γ_q

$$\Gamma_q := \{\sigma 0^\infty : \sigma = \sigma_1 \cdots \sigma_k \in \Sigma_q^*, \sigma_k \neq 0\}. \quad (2.4)$$

Γ_q will play a special role in our construction.

Suppose that for some word $\sigma = \sigma' 0^\infty \in \Gamma_q$, $\tau(\sigma|_k) = 0$ for all k sufficiently large, we say that τ is *regular on σ* , otherwise *irregular*. Let b be another integer, if τ is regular on some $\sigma \in \Gamma_q$, we define the projection Π_b^τ from Γ_q to \mathbb{R} as

$$\Pi_b^\tau(\sigma) = \sum_{k=1}^{\infty} \tau(\sigma|_k) b^{k-1}. \quad (2.5)$$

The above sum is finite since $\tau(\sigma|_k) = 0$ for sufficiently large k . If τ is regular on any σ in Γ_q , we say that τ is a *regular mapping*.

Example 2.3. Suppose $b = q$, let $\mathcal{C} = \{c_0 = 0, c_1, \dots, c_{b-1}\}$ be a residue system mod b where $c_i \equiv i \pmod{b}$. Define $\tau(\vartheta) = 0$ and $\tau(\sigma) = c_{\sigma_k}$ if $\sigma = \sigma_1 \cdots \sigma_k \in \Sigma_q^k \subset \Sigma_q^*$. Then it is easy to see that τ is regular on any $\sigma \in \Gamma_q$ and hence it is regular. Moreover,

$$\Pi_b^\tau(\Gamma_b) \subseteq \mathbb{Z}. \quad (2.6)$$

When $\mathcal{C} = \{0, 1, \dots, b-1\}$, then the mapping Π_b^τ is a bijection from Γ_b onto $\mathbb{N} \cup \{0\}$.

In [DHS], putting their setup in our language, they classified maximal orthogonal sets of standard one-fourth Cantor measure via the mapping τ from Σ_2^* to $\{0, 1, 2, 3\}$. However, some maximal orthogonal sets may have negative elements in which those elements cannot be expressed finitely in 4-adic expansions using digits $\{0, 1, 2, 3\}$. In our classification, we will choose the digit system to be $\mathcal{C} = \{-1, 0, 1, \dots, b-2\}$ in which we can expand any integers uniquely by finite b -adic expansion. We have the following simple but important lemma.

Lemma 2.4. Let $\mathcal{C} = \{-1, 0, 1, \dots, b-2\}$ with integer $b \geq 3$ and let τ be the map defined in Example 2.3. Then Π_b^τ is a bijection between Γ_b and \mathbb{Z} .

Proof. For any $n \in \mathbb{Z}$ and $|n| < b$, it is easy to see that there exists unique $\sigma \in \Gamma_b$ such that $n = \Pi_b^\tau(\sigma)$. For example, $n = b-1$, then $n = \Pi_b^\tau(\sigma_1 \sigma_2)$ where $\sigma_1 = b-1$ and $\sigma_2 = 1$. When $|n| \geq b$, then n can be decomposed uniquely as $n = \ell b + c$ where $c \in \mathcal{C}$. We note that $|\ell| = |\frac{n-c}{b}| \leq \frac{|n|+b-2}{b} < |n|$. If $|\ell| < b$, we are done. Otherwise, we further decompose ℓ in a similar way and after finite number of steps, $|\ell| < b$. The expansion is unique since each decomposition is unique. \square

We now define a q -adic tree labeling which corresponds to a maximal orthogonal set for $\mu_{q,b}$ when $b = qr$. We observe that for $b = qr$, we can decompose $\mathcal{C} = \{-1, 0, \dots, b-2\}$ in q disjoint classes according to the remainders after being divided by q : $\mathcal{C} = \bigcup_{i=0}^{q-1} \mathcal{C}_i$ where

$$\mathcal{C}_i = (i + q\mathbb{Z}) \cap \mathcal{C}.$$

Definition 2.5. Let Σ_q^* be a q -adic tree and $b = qr$, we say that τ is a *maximal mapping* if it is a map $\tau = \tau_{q,b} : \Sigma_q^* \rightarrow \{-1, 0, \dots, b-2\}$ that satisfies

- (i) $\tau(\vartheta) = \tau(0^n) = 0$ for all $n \geq 1$.
- (ii) For all $k \geq 1$, $\tau(\sigma_1 \cdots \sigma_k) \in \mathcal{C}_{\sigma_k}$.
- (iii) For any word $\sigma \in \Sigma_q^*$, there exists σ' such that τ is regular on $\sigma\sigma'0^\infty \in \Gamma_q$.

We call a tree mapping a *regular mapping* if it satisfies (i) and (ii) in above and is regular on any word in Γ_q . Clearly, regular mappings are maximal.

Given a maximal mapping τ , the following sets will be of our main study in this paper.

$$\Lambda(\tau) := \{\Pi_b^\tau(\sigma) : \sigma \in \Gamma_q, \tau \text{ is regular on } \sigma\}. \quad (2.7)$$

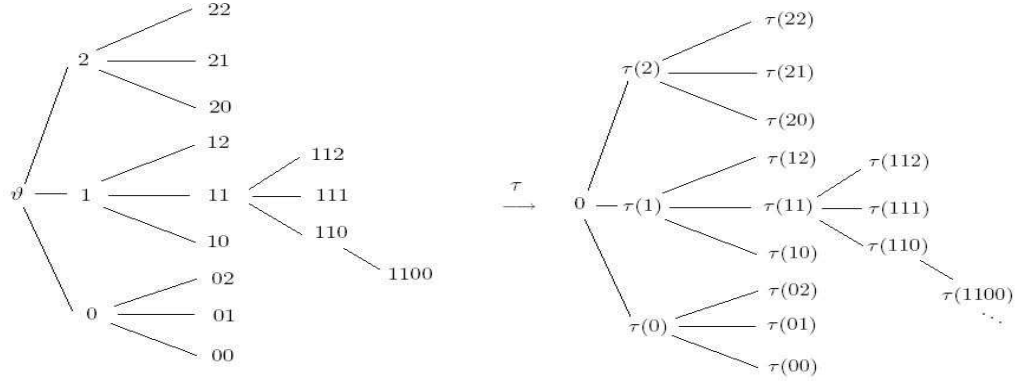


FIGURE 1. An illustration of the 3-adic tree and the associated mapping τ .

From now on, we will assume that $b = qr$, $\mathcal{C} = \{-1, 0, 1, \dots, b-2\}$ and $0 \in \Lambda$. The main results are as follows and this is also the reason why τ is called a maximal mapping.

Theorem 2.6. Λ is a maximal orthogonal set of $L^2(\mu_{q,b})$ if and only if there exists a maximal mapping τ such that $\Lambda = r\Lambda(\tau)$, where $b = qr$.

For the proof, (i) in Definition 2.5 is to ensure $0 \in \Lambda$. (ii) is to make sure the mutually orthogonality and (iii) is for the maximal orthogonality.

If Λ is a spectrum of $L^2(\mu)$, we call the associated maximal mapping τ a *spectral mapping*. We will restrict our attention to regular mappings (i.e. for all $\sigma \in \Gamma_q$, τ is regular on σ). In this case, $\Lambda(\tau) = \{\Pi_b^\tau(\sigma) : \sigma \in \Gamma_q\}$. The advantage of considering regular mappings is that we can give a natural ordering of the maximal orthogonal set $\Lambda(\tau)$. The ordering goes as follows: Given any $n \in \mathbb{N}$, we can expand it into the unique finite q -adic expansion,

$$n = \sum_{j=1}^k \sigma_j q^{j-1}, \quad \sigma_j \in \{0, \dots, q-1\}, \quad \sigma_k \neq 0. \quad (2.8)$$

In this way n is uniquely corresponding to one word $\sigma = \sigma_1 \cdots \sigma_k$, which is called the *q -adic expansion of n* . For a regular mapping τ , there is a natural ordering of the maximal orthogonal set $\Lambda(\tau)$: $\lambda_0 = 0$ and

$$\lambda_n = \Pi_b^\tau(\sigma 0^\infty) = \sum_{j=1}^k \tau(\sigma|_j) b^{j-1} + \sum_{i=k+1}^{N_n} \tau(\sigma 0^{i-k}) b^{i-1} \quad (2.9)$$

where $\sigma = \sigma_1 \cdots \sigma_k$ is the q -adic expansion of n in (2.8), $\tau(\sigma 0^{N_n-k}) \neq 0$ and $\tau(\sigma 0^n) = 0$ for all $n > N_n - k$. Under this ordering, we have $\Lambda(\tau) = \{\lambda_n\}_{n=0}^\infty$. Let

$$N_n^* = \#\{\ell : \sigma = \sigma_1 \cdots \sigma_k, \tau(\sigma 0^\ell) \neq 0\},$$

where we denote $\#A$ the cardinality of the set A . The growth rate of N_n^* is crucial in determining whether $r\Lambda(\tau)$ is a spectrum of $L^2(\mu)$. To describe the growth rate, we let $\mathcal{N}_{m,n}^* = \max\{N_k^* : q^m \leq k < q^n\}$, $\mathcal{L}_n^* = \min\{N_k^* : q^n \leq k < q^{n+1}\}$ and $\mathcal{M}_n = \max\{N_k : 1 \leq k < q^n\}$. We have the following two criteria depending on the growth rate of N_n^* .

Theorem 2.7. *Let $\Lambda = r\Lambda(\tau)$ for a regular mapping τ . Then we can find $0 < c_1 < c_2 < 1$ so that the following holds.*

(i) *If there exists a strictly increasing sequence α_n satisfying*

$$\alpha_{n+1} - \mathcal{M}_{\alpha_n} \rightarrow \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} c_1^{\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*} = \infty, \quad (2.10)$$

then Λ is a spectrum of $L^2(\mu)$.

(ii). *If $\sum_{n=1}^{\infty} c_2^{\mathcal{L}_n^*} < \infty$, then Λ is not a spectrum of $L^2(\mu)$.*

The following is the most important example of Theorem 2.7.

Example 2.8. For a regular mapping τ , if $M := \sup_n \{N_n^*\}$ is finite, then Λ must be a spectrum.

Proof. Note that $\mathcal{N}_{m,n}^* \leq M$ and therefore any strictly increasing sequences α_n will satisfy the second condition in (2.10). Let $\alpha_1 = 1$ and $\alpha_{n+1} = n + \mathcal{M}_{\alpha_n}$ for $n \geq 1$. Then the first condition holds and hence Λ must be a spectrum by Theorem 2.7. \square

We will also see that when there is some slow growth in \mathcal{N}_n^* , Λ can still be a spectrum (see Example 4.4). The exact growth rate for Λ to be a spectrum is however hard to obtain from the techniques we used.

Now, we can construct some spectra which can have zero Beurling dimension from regular orthogonal sets using Theorem 2.7. In fact, they can even be arbitrarily sparse.

Theorem 2.9. *Let $\mu = \mu_{q,b}$ be a measure defined in (1.1) with $b > q$ and $\gcd(q, b) = q$. Then given any increasing non-negative function g on $[0, \infty)$, there exists a spectrum Λ of $L^2(\mu)$ such that*

$$\lim_{R \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{\#(\Lambda \cap (x - R, x + R))}{g(R)} = 0. \quad (2.11)$$

We also make a study on the irregular spectra, although most interesting cases are from the regular one. Let τ be a maximal mapping such that it is irregular on $\{I_1 0^\infty, \dots, I_N 0^\infty\}$, where $I_i \in \Sigma^*$ and the last word in I_i is non-zero, and is regular on the others in Γ_q . We define the corresponding *regularized mapping* τ_R :

$$\tau_R(\sigma) = \begin{cases} 0, & \text{if } \sigma = I_i 0^k \text{ for } k \geq 1; \\ \tau(\sigma), & \text{otherwise.} \end{cases}$$

Our result is as follows:

Theorem 2.10. *Let τ be an irregular maximal mapping of μ . Suppose τ is irregular only on finitely many σ in Γ_q . Then τ is a spectral mapping if and only if a corresponding regularized mapping τ_R is a spectral mapping.*

We will prove this theorem more generally in Theorem 5.1 by showing the spectral property is not affected if we alter only finitely many elements in Γ_q . However, we don't know whether the same holds if the finiteness assumption on irregular elements is removed.

3. MAXIMAL ORTHOGONAL SETS

In this section, we discuss the existence of orthogonal sets for $\mu_{q,b}$, in particular, Theorem 2.1 and Theorem 2.6 are proved.

Proof of Theorem 2.1. Let $\gcd(q, b) = d$. Suppose q and b are relatively prime i.e. $d = 1$. Let

$$\mathcal{Z}_n := \left\{ \frac{b^n}{q} a : q \nmid a \right\}.$$

It is easy to see that $\mathcal{Z}(\widehat{\mu}_{q,b}) = \bigcup_{n=1}^{\infty} \mathcal{Z}_n$. Note that for any a with $q \nmid a$, we have $q \nmid ba$ since $\gcd(q, b) = 1$. Hence, if $n > 1$,

$$\frac{b^n}{q}a = \frac{b^{n-1}}{q}(ba) \in \mathcal{Z}_{n-1}.$$

This implies that $\mathcal{Z}_1 \supset \mathcal{Z}_2 \supset \cdots$ and $\mathcal{Z}(\widehat{\mu}_{q,b}) = \mathcal{Z}_1$. Let

$$Y_i = \left\{ \frac{b}{q}a : q \nmid a, a \equiv i \pmod{q} \right\},$$

then $\mathcal{Z}(\widehat{\mu}_{q,b}) = \bigcup_{i=1}^{q-1} Y_i$. If there exists a mutually orthogonal set Λ for $\mu_{q,b}$ with $\#\Lambda \geq q$, we may assume $0 \in \Lambda$ so that $\Lambda \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{q,b})$. Hence there exists $1 \leq i \leq q-1$ such that $Y_i \cap \Lambda$ contains more than 1 elements, say λ_1, λ_2 . But then $\lambda_1 - \lambda_2 = \frac{b}{q}r$ where $q|r$. This contradicts the orthogonal property of Λ .

Suppose now $d > 1$, we know $d \leq q$. We first consider $d = q$ and prove that the measure is a spectral measure. This shows also the second statement. Write now $b = qr$ and define $\mathcal{D} = \{0, 1, \dots, q-1\}$ and $\mathcal{S} = \{0, r, \dots, (q-1)r\}$. Then it is easy to see that the matrix

$$H := [e^{2\pi i \frac{ijr}{b}}]_{0 \leq i, j \leq q-1} = [e^{2\pi i \frac{ij}{q}}]_{0 \leq i, j \leq q-1}$$

is a Hadamard matrix (i.e. $HH^* = qI$). This shows $\frac{1}{b}\mathcal{D}$ and \mathcal{S} forms a compatible pair as in [LaW]. Therefore it is a spectral measure by Theorem 1.2 in [LaW].

Suppose now $1 < d < q$. We have shown that $\mu_{d,b}$ is a spectral measure and hence $\mathcal{Z}(\widehat{\mu}_{d,b})$ contains an infinite bi-zero sets Λ (i.e. $\Lambda - \Lambda \subset \mathcal{Z}(\widehat{\mu}_{d,b}) \cup \{0\}$). We claim that $\mathcal{Z}(\widehat{\mu}_{d,b}) \subset \mathcal{Z}(\widehat{\mu}_{q,b})$ and hence $\mathcal{Z}(\widehat{\mu}_{q,b})$ has infinitely many orthogonal exponentials. To justify the claim, we write $q = dt$. Note that for $d \nmid a$,

$$\frac{b^n}{d}a = \frac{b^n}{q}(ta).$$

As q cannot divide ta . Hence, $\frac{b^n}{q}(ta) \in \mathcal{Z}(\widehat{\mu}_{q,b})$. This also completes the proof of Theorem 2.1. \square

Remark 3.1. In view of Theorem 2.1, we cannot decide whether there are spectral measures when $1 < \gcd(q, b) < q$. In general, $\mu_{q,b}$ is the convolutions of several self-similar measures with some are spectral and some are not spectral. If $q = 4$, $b = 6$, we know that $\{0, 1, 2, 3\} = \{0, 1\} \oplus \{0, 2\}$ and hence

$$\widehat{\mu}_{4,6}(\xi) = \prod_{j=1}^{\infty} \left(\frac{1 + e^{2\pi i 6^{-j}\xi}}{2} \right) \cdot \prod_{j=1}^{\infty} \left(\frac{1 + e^{2\pi i 26^{-j}\xi}}{2} \right) = \widehat{\nu}_1(\xi) \widehat{\nu}_2(\xi)$$

where $\nu_1 = \mu_{2,6}$ and ν_2 is the equal weight self-similar measure defined by the IFS with maps $\frac{1}{6}x$ and $\frac{1}{6}(x+2)$. Hence, $\mu_{4,6} = \nu_1 * \nu_2$. It is known that both ν_1 and ν_2 are spectral measures, but we don't know whether $\mu_{4,6}$ is a spectral measure. If $q = 6$ and $b = 10$, then $\{0, 1, \dots, 5\} = \{0, 1\} \oplus \{0, 2, 4\}$ and hence $\mu_{6,10}$ is

the convolution of $\mu_{2,10}$ with a non-spectral measure with 3 digits and contraction ratio $1/10$. Because of its convolutional structure, it may be a good testing ground for studying the Laba-Wang conjecture [LaW] and also for finding non-spectral measures with Fourier frame [DL, HLL].

Proof of Theorem 2.6. Suppose $\Lambda = r\Lambda(\tau)$ for some maximal mapping τ . We show that it is an maximal orthogonal set for $L^2(\mu)$. To see this, we first show Λ is a bi-zero set. Pick $\lambda, \lambda' \in \Lambda$, by the definition of $\Lambda(\tau)$, we can find two distinct σ, σ' in Γ_q such that

$$\lambda = \frac{b}{q}\Pi_b^\tau(\sigma), \quad \lambda' = \frac{b}{q}\Pi_b^\tau(\sigma').$$

Let k be the first index such that $\sigma|_k \neq \sigma'|_k$. Then for some integer M , we can write

$$q\lambda - q\lambda' = b \sum_{i=k}^{\infty} (\tau(\sigma|_i) - \tau(\sigma'|_i))b^{i-1} = b^k ((\tau(\sigma|_k) - \tau(\sigma'|_k)) + bM).$$

By (ii) in Definition 2.5, $\tau(\sigma|_k)$ and $\tau(\sigma'|_k)$ are in distinct residue class of q . This means q does not divide $\tau(\sigma|_k) - \tau(\sigma'|_k)$. On the other hand, q divides b . Hence, q does not divide $(\tau(\sigma|_k) - \tau(\sigma'|_k)) + bM$. By (2.3), $\lambda - \lambda'$ lies in $\mathcal{Z}(\hat{\mu})$.

To establish the maximality of the orthogonal set Λ , we show by contradiction. Let $\theta \notin \Lambda$ but θ is orthogonal to all elements in Λ . Since $0 \in \Lambda$, $\theta \neq 0$ and $\theta = \theta - 0 \in \mathcal{Z}(\hat{\mu})$. Hence, by (2.3) we may write

$$\theta = r(b^{k-1}a),$$

where q does not divide a . Expand $b^{k-1}a$ in b -adic expansion using digits $\{-1, 0, \dots, b-2\}$

$$b^{k-1}a = \epsilon_{k-1}b^{k-1} + \epsilon_k b^k + \dots + \epsilon_{k+\ell} b^{k+\ell},$$

q does not divide ϵ_{k-1} . Note that there exists unique σ_s , $0 \leq \sigma_s \leq q-1$, such that $\epsilon_s \equiv \sigma_s \pmod{q}$ for $k-1 \leq s \leq k+\ell$. Denote $\sigma_s = \epsilon_s = 0$ for $s > k+\ell$. Since $\theta \notin \Lambda$, we can find the smallest integer α such that $\tau(0^{k-2}\sigma_{k-1}\sigma_k \dots \sigma_\alpha) \neq \epsilon_\alpha$. By (iii) in the definition of τ , we can find $\sigma \in \Gamma_q$ so that $\sigma = 0^{k-2}\sigma_{k-1}\sigma_k \dots \sigma_\alpha \sigma' 0^\infty$ and τ is regular on σ , then there exists M' such that

$$\theta - r\Pi_b^\tau(\sigma) = rb^\alpha(\epsilon_\alpha - \tau(0^{k-1}\sigma_{k-1} \dots \sigma_\alpha) + M'b).$$

By (ii) in the definition of τ , $\tau(0^{k-1}\sigma_{k-1} \dots \sigma_\alpha) \equiv \sigma_\alpha \pmod{q}$, which is also congruent to ϵ_α by our construction. This implies $\theta - r\Pi_b^\tau(\sigma)$ is not in the zero set of $\hat{\mu}$ since q divides $\epsilon_\alpha - \tau(0^{k-1}\sigma_{k-1} \dots \sigma_\alpha)$ and b does not divide it either. It contradicts to θ being orthogonal to all Λ .

Conversely, suppose we are given a maximal orthogonal set Λ of $L^2(\mu)$ with $0 \in \Lambda$. Then $\Lambda \subset \mathcal{Z}(\hat{\mu})$. Hence, we can write

$$\Lambda = \{ra_\lambda : \lambda \in \Lambda, a_\lambda = b^{k-1}m \text{ for some } k \geq 1 \text{ and } m \in \mathbb{Z} \text{ with } q \nmid m\},$$

where $a_0 = 0$. Now, expand a_λ in b -adic expansion with digits chosen from $\mathcal{C} = \{-1, 0, \dots, b-2\}$.

$$a_\lambda = \sum_{j=1}^{\infty} \epsilon_\lambda^{(j)} b^{j-1}. \quad (3.1)$$

Let $D(\vartheta) = \{\epsilon_\lambda^{(1)} : \lambda \in \Lambda\}$ be all the first coefficients of b -adic expressions (3.1) of elements in Λ , and let $D(c_1, \dots, c_n) = \{\epsilon_\lambda^{(n+1)} : \epsilon_\lambda^{(1)} = c_1, \dots, \epsilon_\lambda^{(n)} = c_n, \lambda \in \Lambda\}$ be all the $n+1$ -st coefficients of elements in Λ whose first n coefficients are fixed, where $c_1, c_2, \dots, c_n \in \mathcal{C}$. We need the following lemma.

Lemma 3.2. *With the notations above, then $D(\vartheta)$ contains exactly q elements which are in distinct residue class (mod q) and $0 \in D(\vartheta)$. Moreover, if $D(c_1, \dots, c_n)$ with all $c_i \in \mathcal{C}$ is non-empty, then it contains exactly q elements which are in distinct residue class (mod q) also. In particular, $0 \in D(c_1, \dots, c_n)$ if $c_1 = \dots = c_n = 0$ for $n \geq 1$.*

Proof. Clearly, by (3.1), $0 \in D(\vartheta)$ and $0 \in D(c_1, \dots, c_n)$ if $c_1 = \dots = c_n = 0$ for $n \geq 1$. Suppose the number of elements in $D(\vartheta)$ is strictly less than q . We let $\alpha \in \mathcal{C} \setminus D(\vartheta)$ such that α is not congruent to any elements in $D(\vartheta)$. Then, for any $\lambda \in \Lambda$, by (3.1) we have

$$r\alpha - \lambda = r \left(\alpha - \sum_{j=1}^{\infty} \epsilon_\lambda^{(j)} b^{j-1} \right) = \frac{b}{q} \left(\alpha - \epsilon_\lambda^{(1)} + \sum_{j=2}^{\infty} \epsilon_\lambda^{(j)} b^{j-1} \right).$$

Note that $q \nmid (\alpha - \epsilon_\lambda^{(1)})$ for all $\lambda \in \Lambda$ by the assumption, this implies $r\alpha$ is mutually orthogonal to Λ but is not in Λ , which contradicts to maximal orthogonality. Hence $D(\vartheta)$ contains at least q elements. If $D(\vartheta)$ contains more than q elements, then there exists $a_{\lambda_1} = \sum_{j=1}^{\infty} \epsilon_{\lambda_1}^{(j)} b^{j-1}, a_{\lambda_2} = \sum_{j=1}^{\infty} \epsilon_{\lambda_2}^{(j)} b^{j-1}$ such that $\epsilon_{\lambda_1}^{(1)} \equiv \epsilon_{\lambda_2}^{(1)} \pmod{q}$ and $\epsilon_{\lambda_1}^{(1)} \neq \epsilon_{\lambda_2}^{(1)}$. Then $r(a_{\lambda_1} - a_{\lambda_2}) = \frac{b}{q}(\epsilon_{\lambda_1}^{(1)} - \epsilon_{\lambda_2}^{(1)} + bM)$ for some integer M . This means $r(a_{\lambda_1} - a_{\lambda_2})$ is not a zero of $\widehat{\mu}$. This contradicts to the mutually orthogonality. Hence, $D(\vartheta)$ contains exactly q elements which are in distinct residue class (mod q).

In general, we proceed by induction. Suppose the statement holds up to $n-1$. If now $D(c_1, \dots, c_n)$ is non-empty, then $D(c_1, \dots, c_k)$ is also non-empty for all $k \leq n$. we now show that $D(c_1, \dots, c_n)$ must contain at least q elements. Otherwise, we consider $\theta = r(c_1 + \dots + c_n b^{n-1} + \alpha b^n)$ where α is in $\mathcal{C} \setminus D(c_1, \dots, c_n)$ and α is not congruent to any elements in $D(c_1, \dots, c_n) \pmod{q}$. If $\lambda \in \Lambda$ and $\lambda = r(c_1 + \dots + c_k b^{k-1} + c'_{k+1} b^k + \dots)$ where $c_{k+1} \neq c'_{k+1}$ and $k \leq n$, then $c_{k+1}, c'_{k+1} \in D(c_1, \dots, c_k)$ and hence θ and λ are mutually orthogonal by the induction hypothesis. If $\lambda \in \Lambda$ is such that the first n digit expansion are equal to θ , the same argument as in the proof of $D(\vartheta)$ shows θ will be orthogonal to this λ . Therefore, θ will be orthogonal to

all elements in Λ , a contradiction. Also in a similar way as the above, $D(c_1, \dots, c_n)$ contains exactly q elements can be shown. \square

Returning to the proof, by convention, we define $\tau(\vartheta) = 0$ and on the first level, we define $\tau(\sigma_1)$ to be the unique element in $D(\vartheta)$ such that it is congruent to $\sigma_1 \pmod{q}$. For $\sigma = \sigma_1 \cdots \sigma_n$, we define $\tau(\sigma\sigma_{n+1})$ to be the unique element in $D(\tau(\sigma|_1), \dots, \tau(\sigma|_n))$ (it is non-empty from the induction process) that is congruent to $\sigma_{n+1} \pmod{q}$. Then $\tau(0^k) = 0$ for $k \geq 1$.

We show that τ is a maximal mapping corresponding to Λ . (i) is satisfied by above. By Lemma 3.2, τ is well-defined with (ii) in Definition 2.5. Finally, given a node $\sigma \in \Sigma_q^n$, by the construction of the τ we can find λ whose first n digits in the digit expansion (3.1) exactly equals the value of $\tau(\sigma|_k)$ for all $1 \leq k \leq n$. Since the digit expansion of λ becomes 0 eventually, we continue following the digit expansion of λ so that (iii) in the definition is satisfied.

We now show that $\Lambda = r\Lambda(\tau)$. For each a_λ given in (3.1), Lemma 3.2 with the definition of τ shows that there exists unique path σ such that $\tau(\sigma|_n) = \epsilon_\lambda^{(n)}$ for all n . As the sum is finite, this means $\Lambda \subset r\Lambda(\tau)$. Conversely, if some $\Pi_b^r(\sigma) \in r\Lambda(\tau)$ is not in Λ , then from the previous proof we know $\Pi_b^r(\sigma)$ must be orthogonal to all elements in Λ . This contradicts to the maximal orthogonality of Λ . Thus, $\Lambda = r\Lambda(\tau)$.

4. REGULAR SPECTRA

In the rest of the paper, we study under what conditions a maximal orthogonal set is a spectrum or not a spectrum.

Lemma 4.1. [JP] *Let μ be a Borel probability measure in \mathbb{R} with compact support. Then a countable set Λ is a spectrum for $L^2(\mu)$ if and only if*

$$Q(\xi) := \sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi + \lambda)|^2 \equiv 1, \quad \text{for } \xi \in \mathbb{R}.$$

Moreover, if Λ is a bi-zero set, then Q is an entire function.

Note that the first part of Lemma 4.1 is well-known. For the entire function property, we just note that the partial sum $\sum_{|\lambda| \leq n} |\cdots|^2$ is an entire function and it is locally uniformly bounded by applying the Bessel's inequality, hence Q is entire by the Montel's theorem in complex analysis. One may refer to [JP] for the details of the proof.

Let δ_a be the Dirac measure with center a . We define

$$\delta_{\mathcal{E}} = \frac{1}{\#\mathcal{E}} \sum_{e \in \mathcal{E}} \delta_e$$

for any finite set \mathcal{E} . Let μ be the self-similar measure in (1.1). Write $\mathcal{D} = \{0, 1, \dots, q-1\}$ and $D_k = \frac{1}{b}\mathcal{D} + \cdots + \frac{1}{b^k}\mathcal{D}$ for $k \geq 1$. We recall that the *mask*

function of \mathcal{D} is

$$m(\xi) = \frac{1}{q}(1 + e^{2\pi i \xi} + \dots + e^{2\pi i(q-1)\xi})$$

and define $\mu_k = \delta_{\mathcal{D}_k}$, then

$$\widehat{\mu_k}(\xi) = \prod_{j=1}^k m(b^{-j}\xi)$$

and it is well-known that μ_k converges weakly to μ when k tends to infinity and we have

$$\widehat{\mu}(\xi) = \widehat{\mu_k}(\xi) \widehat{\mu}\left(\frac{\xi}{b^k}\right). \quad (4.1)$$

Lemma 4.2. *Let τ be a regular mapping and let $\Lambda = r\Lambda(\tau) = \{\lambda_k\}_{k=0}^\infty$ be the maximal orthogonal set determined by τ . Then for all $n \geq 1$,*

$$\sum_{k=0}^{q^n-1} |\widehat{\mu_n}(\xi + r\lambda_k)|^2 \equiv 1. \quad (4.2)$$

Proof. We claim that $\{r\lambda_k\}_{k=0}^{q^n-1} = \frac{b}{q}\{\lambda_k\}_{k=0}^{q^n-1}$ is a spectrum of $L^2(\mu_n)$. We can then use Lemma 4.1 to conclude our lemma. Since this set has exactly q^n elements, we just need to show the mutually orthogonality. To see this, we note that

$$\widehat{\mu_n}(\xi) = m\left(\frac{\xi}{b}\right) \cdots m\left(\frac{\xi}{b^n}\right). \quad (4.3)$$

Given $l \neq l'$ in $\{0, \dots, q^n - 1\}$, let $\sigma = \sigma_1 \cdots \sigma_n$ and $\sigma' = \sigma'_1 \cdots \sigma'_n$ be the q -adic expansions of l and l' respectively as in (2.8), where σ_n and σ'_n may be zero. We let $s \leq n$ be the first index such that $\sigma_s \neq \sigma'_s$. Then we can write

$$\lambda_l - \lambda_{l'} = b^{s-1}(\tau(\sigma|_s) - \tau(\sigma'|_s) + bM)$$

for some integer M . We then have from the integral periodicity of $e^{2\pi i x}$ that

$$m\left(\frac{r(\lambda_l - \lambda_{l'})}{b^s}\right) = m\left(\frac{\tau(\sigma|_s) - \tau(\sigma'|_s)}{q}\right) = 0.$$

It is equal to 0 because (ii) in the definition of maximal mapping implies that q does not divide $\tau(\sigma|_s) - \tau(\sigma'|_s)$. Hence, by (4.3), $\widehat{\mu_n}(r(\lambda_l - \lambda_{l'})) = 0$. \square

Now, we let

$$Q_n(\xi) = \sum_{k=0}^{q^n-1} |\widehat{\mu}(\xi + r\lambda_k)|^2, \text{ and } Q(\xi) = \sum_{k=0}^{\infty} |\widehat{\mu}(\xi + r\lambda_k)|^2.$$

For any n and p , we have the following identity:

$$\begin{aligned} Q_{n+p}(\xi) &= Q_n(\xi) + \sum_{k=q^n}^{q^{n+p}-1} |\widehat{\mu}(\xi + r\lambda_k)|^2 \\ &= Q_n(\xi) + \sum_{k=q^n}^{q^{n+p}-1} |\widehat{\mu_{n+p}}(\xi + r\lambda_k)|^2 \left| \widehat{\mu}\left(\frac{\xi + r\lambda_k}{b^{n+p}}\right) \right|^2. \end{aligned} \quad (4.4)$$

Our goal is see whether $Q(\xi) \equiv 1$. Then by invoking Lemma 4.1, we can determine whether we have a spectrum. As Q is an entire function by Lemma 4.1, we just need to see the value of $Q(\xi)$ for some small values of ξ . To do this, we need to make a fine estimation of the terms $|\widehat{\mu}(\frac{\xi+r\lambda_k}{b^{n+p}})|^2$ in the above. Write

$$c_{\min} = \min \left\{ \prod_{j=0}^{\infty} |m(b^{-j}\xi)|^2 : |\xi| \leq \frac{b-1}{qb} \right\} > 0,$$

where $|m(\xi)| = \frac{|\sin \pi q \xi|}{q |\sin \pi \xi|}$ and $\prod_{j=0}^{\infty} |m(b^{-j}\xi)|^2 = |m(\xi)\widehat{\mu}(\xi)|^2$. Denote

$$c_{\max} = \max \left\{ |m(\xi)|^2 : \frac{1}{b^2} \leq |\xi| \leq \frac{b-1}{qb} \right\} < 1.$$

The following proposition roughly says that the magnitude of the Fourier transform is controlled by the number of non-zero digits in the b -adic expansion in a uniform way. Recall that $b = qr$ with $q, r \geq 2$.

Proposition 4.3. *Let $|\xi| \leq \frac{r(b-2)}{b-1}$ and let*

$$t = \xi + \sum_{k=1}^N d_k b^{n_k},$$

where $d_i \in \{1, 2, \dots, r-1\}$ and $1 \leq n_1 < \dots < n_N$. Then

$$c_{\min}^{N+1} \leq |\widehat{\mu}(t)|^2 \leq c_{\max}^N. \quad (4.5)$$

Proof. First it is easy to check that, for $|\xi| \leq \frac{r(b-2)}{b-1}$ and all $d_k \in \{0, 1, 2, \dots, r-1\}$, we have

$$\begin{aligned} \left| \frac{\xi + \sum_{k=1}^n d_k b^k}{b^{n+1}} \right| &\leq \frac{1}{b^{n+1}} \left(\frac{r(b-2)}{b-1} + (r-1)(b + b^2 + \dots + b^n) \right) \\ &= \frac{r(b-2) + (r-1)(b^{n+1} - b)}{b^{n+1}(b-1)} \\ &\leq \frac{b-1}{qb} \end{aligned} \quad (4.6)$$

for $n \geq 1$. The inequality in the last line follows from a direct comparison of the difference and $q \geq 2$. To simplify notations, we let $n_0 = 0$ and $n_{N+1} = \infty$. Then $|\widehat{\mu}(t)|^2$ equals

$$\prod_{j=1}^{\infty} |m(b^{-j}t)|^2 = \prod_{i=0}^N \prod_{j=n_i+1}^{n_{i+1}} |m(b^{-j}t)|^2. \quad (4.7)$$

We now estimate the products one by one. By (4.6), we have

$$\left| \frac{\xi + \sum_{k=1}^i d_k b^{n_k}}{b^{n_i+1}} \right| \leq \frac{b-1}{qb}.$$

Hence, together with the integral periodicity of m and the definition of c_{\min} , we have for all $i > 0$,

$$\begin{aligned} \prod_{j=n_i+1}^{n_{i+1}} |m(b^{-j}t)|^2 &= \prod_{j=n_i+1}^{n_{i+1}} \left| m \left(b^{-j} \left(\xi + \sum_{k=1}^i d_k b^{n_k} \right) \right) \right|^2 \\ &\geq \prod_{j=0}^{\infty} \left| m \left(b^{-j} \left(\frac{\xi + \sum_{k=1}^i d_k b^{n_k}}{b^{n_i+1}} \right) \right) \right|^2 \geq c_{\min}. \end{aligned} \quad (4.8)$$

For the case $i = 0$, it is easy to see that $\left| \frac{\xi}{b} \right| \leq \frac{b-2}{q(b-1)} < \frac{b-1}{qb}$. Hence, $\prod_{j=n_0+1}^{n_1} |m(b^{-j}t)|^2 \geq \prod_{j=0}^{\infty} |m(b^{-j}(\xi/b))|^2 \geq c_{\min}$. Putting this fact and (4.8) into (4.7), we have $|\widehat{\mu}(t)|^2 \geq c_{\min}^{N+1}$.

We next prove the upper bound. From $|m(\xi)| \leq 1$, (4.7) and the integral periodicity of m ,

$$|\widehat{\mu}(t)|^2 \leq \prod_{i=1}^N |m(b^{-(n_i+1)}t)|^2 = \prod_{i=1}^N \left| m \left(b^{-(n_i+1)} \left(\xi + \sum_{k=1}^i d_k b^{n_k} \right) \right) \right|^2. \quad (4.9)$$

By (4.6) we have

$$\left| \xi + \sum_{k=1}^i d_k b^{n_k} \right| \geq b^{n_i} - \left| \xi + \sum_{k=1}^{i-1} d_k b^{n_k} \right| \geq b^{n_i} - \frac{b^{n_{i-1}}(b-1)}{q} \geq b^{n_i-1}.$$

By (4.6), (4.9), the above and the definition of c_{\max} , we obtain that $|\widehat{\mu}(t)|^2 \leq c_{\max}^N$. \square

We now prove Theorem 2.7. Write $c_1 = c_{\min}$ and $c_2 = c_{\max}$, where c_{\min} and c_{\max} are in Proposition 4.3. Also recall the quantities defined in Section 2. For any $n \in \mathbb{N}$, the q -adic expression of n is $\sum_{j=1}^k \sigma_j q^{j-1}$ with $\sigma_k \neq 0$. Then for the map τ we have

$$\lambda_n = \sum_{j=1}^k \tau(\sigma_1 \cdots \sigma_j) b^{j-1} + \sum_{j=k+1}^{N_n} \tau(\sigma_1 \cdots \sigma_k 0^{j-k}) b^{j-1}$$

where $\tau(\sigma_1 \cdots \sigma_k 0^{N_n-k}) \neq 0$ and $N_n^* = \#\{\tau(\sigma_1 \cdots \sigma_k 0^j) \neq 0 : k+1 \leq j \leq N_n\}$. Moreover, $\mathcal{N}_{m,n}^* = \max_{q^m \leq k < q^n} \{N_k^*\}$, $\mathcal{L}_n^* = \min_{q^n \leq k < q^{n+1}} \{N_k^*\}$ and $\mathcal{M}_n = \max_{1 \leq k < q^n} \{N_k\}$.

Proof of Theorem 2.7. (i) Let α_n be the increasing sequence satisfying (2.10) and let $|\xi| \leq \frac{b-2}{b-1}$. Recall (4.4),

$$Q_{\alpha_{n+1}}(q^{-1}\xi) = Q_{\alpha_n}(q^{-1}\xi) + \sum_{k=q^{\alpha_n}}^{q^{\alpha_{n+1}}-1} \left| \widehat{\mu}_{\alpha_{n+1}}(q^{-1}\xi + r\lambda_k) \right|^2 \left| \widehat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}}\right) \right|^2.$$

For $k = q^{\alpha_n}, \dots, q^{\alpha_{n+1}} - 1$, we may write λ_k as

$$\lambda_k = \sum_{j=0}^{\alpha_{n+1}-1} c_j b^j + \sum_{j=1}^{M_k} d_j q b^{n_j},$$

where $c_j \in \{-1, \dots, b-2\}$, $d_j \in \{1, \dots, r-1\}$ and $\alpha_{n+1} \leq n_1 < n_2 < \dots < n_{M_k}$ with $n_{M_k} = N_k$ and $M_k \leq N_k^*$, where N_k, N_k^* was defined in (2.9) or see the above. Note also that the second term on the right hand of the above is zero whenever $N_k < \alpha_{n+1}$. Now,

$$\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}} = \frac{q^{-1}\xi + q^{-1} \sum_{j=1}^{\alpha_{n+1}} c_j b^j}{b^{\alpha_{n+1}}} + \sum_{j=1}^{M_k} d_j b^{n_j - \alpha_{n+1} + 1}.$$

Note that

$$\left| \frac{\xi}{q} + \frac{1}{q} \sum_{j=1}^k c_j b^j \right| \leq \frac{b-2}{q(b-1)} + (b-2) \frac{b^{k+1} - b}{q(b-1)} \leq \frac{b-2}{q(b-1)} b^{k+1} = \frac{r(b-2)}{b-1} b^k$$

for all $k \geq 1$. Hence, Proposition 4.3 implies that

$$\left| \widehat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}}\right) \right|^2 \geq c_1^{1+M_k} \geq c_1^{1+N_k^*} \geq c_1^{1+\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*}$$

for all $q^{\alpha_n} \leq k < q^{\alpha_{n+1}}$. Therefore, together with Lemma 4.2,

$$\begin{aligned} Q_{\alpha_{n+1}}(q^{-1}\xi) &\geq Q_{\alpha_n}(q^{-1}\xi) + c_1^{1+\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*} \sum_{k=q^{\alpha_n}}^{q^{\alpha_{n+1}}-1} \left| \widehat{\mu}_{\alpha_{n+1}}(q^{-1}\xi + r\lambda_k) \right|^2 \\ &= Q_n(q^{-1}\xi) + c_1^{1+\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*} \left(1 - \sum_{k=0}^{q^{\alpha_n}-1} \left| \widehat{\mu}_{\alpha_{n+1}}(q^{-1}\xi + r\lambda_k) \right|^2 \right). \end{aligned} \quad (4.10)$$

From elementary analysis, there exists δ , $0 < \delta < 1$, such that $|\widehat{\mu}(\xi)|^2$ is decreasing on $(0, \delta)$ (In fact, since $\widehat{\mu}(0) = 1$ and $|\widehat{\mu}(\xi)|^2$ is entire in complex plane, there exists $\eta > 0$ such that $|\widehat{\mu}(\xi)| < 1$ for all $0 < \xi < \eta$. If $|\widehat{\mu}(\xi)|^2$ is not decreasing on $(0, \delta)$ for any $\delta > 0$, we can find a sequence $\xi_n \rightarrow 0$ such that $(|\widehat{\mu}|^2)'(\xi_n) = 0$ and thus $(|\widehat{\mu}|^2)' \equiv 0$ by the entire function property of $|\widehat{\mu}|^2$, this is impossible). In the proof, it is also useful to note that $|\widehat{\mu}(-\xi)| = |\widehat{\mu}(\xi)|$. We now argue by contradiction.

Suppose there exists Λ such that Theorem 2.7 (i) holds but is not a spectrum, then there exists $t_0 < \min\{\delta, \frac{b-2}{b-1}\}$ such that $Q(q^{-1}t_0) < 1$ because Q is entire. For $0 \leq k \leq q^{\alpha_n} - 1$, we have

$$\left| \frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}} \right| \leq \frac{1 + rb^{\mathcal{M}_{\alpha_n}}}{b^{\alpha_{n+1}}} := \beta_n.$$

By the assumption that $\alpha_{n+1} - \mathcal{M}_{\alpha_n} \rightarrow \infty$, we have for all n large, say $n \geq M$, $\beta_n < \delta$ so that $\left| \widehat{\mu}\left(\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}\right) \right|^2 \geq |\widehat{\mu}(\beta_n)|^2$ and we can find $r < 1$ such that

$$|\widehat{\mu}(\beta_n)|^{-2} Q(q^{-1}t_0) \leq r < 1, \quad \text{for } n \geq M$$

because β_n tends to zero when n tends to infinity and $\widehat{\mu}(0) = 1$. According to $\widehat{\mu}(\xi) = \widehat{\mu}_{\alpha_{n+1}}(\xi) \widehat{\mu}(\xi/b^{\alpha_{n+1}})$, we have

$$\begin{aligned} |\widehat{\mu}(q^{-1}t_0 + r\lambda_k)|^2 &= \left| \widehat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k) \widehat{\mu}\left(\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}\right) \right|^2 \\ &\geq |\widehat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)|^2 |\widehat{\mu}(\beta_n)|^2 \\ &\geq \frac{Q(q^{-1}t_0)}{r} |\widehat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)|^2 \end{aligned}$$

From (4.10) and for all $n \geq M$,

$$\begin{aligned} Q_{\alpha_{n+1}}(q^{-1}t_0) &\geq Q_{\alpha_n}(q^{-1}t_0) + \left(1 - \frac{r}{Q(q^{-1}t_0)} \sum_{k=0}^{q^{\alpha_n}-1} |\widehat{\mu}(q^{-1}t_0 + r\lambda_k)|^2 \right) c_1^{1+\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*} \\ &\geq Q_{\alpha_n}(q^{-1}t_0) + (1-r)c_1^{1+\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*}. \end{aligned}$$

Taking summation on n from M to $M+p$ where $p > 0$ and note that $Q_n(t) \leq 1$ for any n we have

$$1 \geq Q_{\alpha_{M+p+1}}(q^{-1}t_0) \geq Q_{\alpha_M}(q^{-1}t_0) + (1-r) \sum_{n=M}^{M+p} c_1^{1+\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*}.$$

As $\sum_{n=M}^{\infty} c_1^{\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*} = \infty$ by the assumption, the right hand side of the above tends to infinity. This is impossible. Hence, Λ must be a spectrum.

(ii). The proof starts again at (4.4) with $p = 1$, we have

$$Q_{n+1}(q^{-1}\xi) = Q_n(q^{-1}\xi) + \sum_{k=q^n}^{q^{n+1}-1} \left| \widehat{\mu}_{n+1}(q^{-1}\xi + r\lambda_k) \right|^2 \left| \widehat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{n+1}}\right) \right|^2.$$

Since $N_k^* \geq \mathcal{L}_n^*$ for $q^n \leq k < q^{n+1}$, so Proposition 4.3 implies that

$$Q_{n+1}(q^{-1}\xi) \leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}_n^*} \sum_{k=q^n}^{q^{n+1}-1} \left| \widehat{\mu}_{n+1}(q^{-1}\xi + r\lambda_k) \right|^2.$$

Using Lemma 4.2 and note that $|\widehat{\mu_{n+1}}(\xi)|^2 \geq |\widehat{\mu}(\xi)|^2$, we have

$$\begin{aligned} Q_{n+1}(q^{-1}\xi) &\leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}_n^*} \left(1 - \sum_{k=0}^{q^n-1} |\widehat{\mu_{n+1}}(q^{-1}\xi + r\lambda_k)|^2 \right) \\ &\leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}_n^*} (1 - Q_n(q^{-1}\xi)). \end{aligned}$$

Hence,

$$1 - Q_{n+1}(q^{-1}\xi) \geq (1 - Q_n(q^{-1}\xi))(1 - c_2^{\mathcal{L}_n^*}) \geq \cdots \geq (1 - Q_1(q^{-1}\xi)) \prod_{k=1}^n (1 - c_2^{\mathcal{L}_k^*}) \quad (4.11)$$

Since $\sum_n c_2^{\mathcal{L}_n^*} < \infty$, $B := \prod_{k=1}^{\infty} (1 - c_2^{\mathcal{L}_k^*}) > 0$ and hence as n tends to infinity in (4.11), we have

$$1 - Q(q^{-1}\xi) \geq (1 - Q_1(q^{-1}\xi)) \cdot B > 0.$$

Therefore, τ is not a spectral mapping. \square

As known from Remark 2.8, τ is a spectral mapping if $\sup\{N_n^*\}$ is finite. Now, we give an example of spectrum with slow growth rate of N_n^* .

Example 4.4. Let τ be a regular mapping so that $N_n \leq \log_q n + \log_{c_1^{-2}} \log_q n$ and $N_n^* \leq \log_{c_1^{-2}} \log_q n$ for $n \geq 1$, where c_1 is given in Theorem 2.7. Then $r\Lambda(\tau)$ is a spectrum of $L^2(\mu)$.

Proof. Take $\alpha_n = n^2$. Recall $\mathcal{M}_n = \max_{1 \leq k < q^n} N_k$, we have

$$\alpha_{n+1} - \mathcal{M}_{\alpha_n} \geq (n+1)^2 - n^2 - \log_{c_1^{-2}} n^2,$$

which tends to infinity when n tends to infinity. Note that

$$\mathcal{N}_{\alpha_n, \alpha_{n+1}}^* = \max_{q^{\alpha_n} \leq k < q^{\alpha_{n+1}}} \mathcal{N}_k^* \leq \log_{c_1^{-2}} \log_q q^{(n+1)^2} = \log_{c_1^{-2}} (n+1)^2.$$

Then

$$\sum_{n=1}^{\infty} c_1^{\mathcal{N}_{\alpha_n, \alpha_{n+1}}^*} \geq \sum_{n=1}^{\infty} c_1^{\log_{c_1^{-2}} (n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

By Theorem 2.7 the result follows. \square

On the other hand, if N_n^* is so that $\mathcal{L}_n^* \geq (1+\epsilon) \log_{c_2^{-1}} n$, for some $\epsilon > 0$ and $n \geq 1$, then $r\Lambda(\tau)$ is not a spectrum. This is done by checking the condition of Theorem 2.7(ii) using the similar method in the above. We therefore omit its detail. Finally, we prove Theorem 2.9.

Proof of Theorem 2.9. For any $n \in \mathbb{N}$, n can be expressed as

$$n = \sum_{j=1}^k \sigma_j q^{j-1}, \quad (4.12)$$

where all $\sigma_j \in \{0, 1, \dots, q-1\}$ and $\sigma_k \neq 0$. Let $\{m_n\}_{n=1}^\infty$ be a strictly increasing sequence of positive integers with $m_1 \geq 2$. We now define a maximal mapping in terms of this sequence by $\tau(\vartheta) = \tau(0^k) = 0$ for $k \geq 1$ and for n as in (4.12),

$$\tau(\sigma) = \begin{cases} \sigma_k, & \text{if } \sigma = \sigma_1 \cdots \sigma_k, \sigma_k \neq 0; \\ 0, & \text{if } \sigma = \sigma_1 \cdots \sigma_k 0^\ell, \ell \neq m_n; \\ q, & \text{if } \sigma = \sigma_1 \cdots \sigma_k 0^\ell \text{ and } \ell = m_n. \end{cases}$$

By the definition we have $\lambda_0 = 0$ and

$$\lambda_n = \sum_{j=1}^k \tau(\sigma_1 \cdots \sigma_j) b^{j-1} + q b^{m_n},$$

consequently, $N_n^* = 1$ and by Theorem 2.7(i) (see also Remark 2.8), $\Lambda := \{\lambda_n\}_{n=0}^\infty$ is a spectrum for $L^2(\mu)$.

We now find Λ with density in (2.11) zero by choosing m_n . To do this, we first note that there exists a strictly increasing continuous function $h(t)$ from $[0, \infty)$ onto itself such that $h(t) \leq g(t)$ for $t \geq 0$ and it is sufficient to replace $g(t)$ by $h(t)$ in the proof. In this way, the inverse of $h(t)$ exists, and we denote it by $h^{-1}(t)$.

Now, note that

$$\lambda_n \leq q \frac{b^k - 1}{b - 1} + q b^{m_n} \leq (q + 1) b^{m_n}.$$

Hence,

$$\lambda_{n+1} - \lambda_n \geq q b^{m_{n+1}} - (q + 1) b^{m_n} \geq b^{m_n+1}. \quad (4.13)$$

Therefore, we choose m_n so that $b^{m_n} \geq 2h^{-1}(b^{n+1})$ for all $n \geq 1$. For any $h(R) \geq 1$, there exists unique $s \in \mathbb{N}$ such that $b^{s-1} \leq h(R) < b^s$. Then

$$\frac{\sup_{x \in \mathbb{R}} \#(\Lambda \cap (x - R, x + R))}{h(R)} \leq \frac{\sup_{x \in \mathbb{R}} \#(\Lambda \cap (x - h^{-1}(b^s), x + h^{-1}(b^s)))}{b^{s-1}}.$$

Note from (4.13) that the length of the open intervals $(x - h^{-1}(b^s), x + h^{-1}(b^s))$ is less than $\lambda_{n+1} - \lambda_n$ whenever $n \geq s$. This implies that the set $\Lambda \cap (x - h^{-1}(b^s), x + h^{-1}(b^s))$ contains at most one λ_n where $n \geq s$. We therefore have

$$\sup_{x \in \mathbb{R}} \#(\Lambda \cap (x - h^{-1}(b^s), x + h^{-1}(b^s))) \leq s + 1.$$

Thus the result follows by taking limit. □

5. IRREGULAR SPECTRA

Let τ be a maximal mapping (not necessarily regular) for $\mu = \mu_{q,b}$ with $b = qr$. Given any $I = \sigma_1 \cdots \sigma_k \in \Sigma_q^k = \{0, 1, \dots, q-1\}^k$ with $\sigma_k \neq 0$. Define a map τ' by

$$\tau'(\sigma) = \begin{cases} 0, & \sigma = I0^\ell \text{ for } \ell \geq 1; \\ \tau(\sigma), & \text{otherwise.} \end{cases}$$

Clearly τ' is a maximal mapping. The main result is as follows:

Theorem 5.1. *With the notation above, τ is a spectral mapping if and only if τ' is a spectral mapping.*

This result shows that if we arbitrarily change the value of τ along an element in Γ_q as above, the spectral property of τ is unaffected. In particular, Theorem 2.10 follows as a corollary because we can alter the irregular elements one by one using Theorem 5.1 recursively.

We now prove Theorem 5.1. Note that we can decompose

$$\Gamma_q = \{\sigma 0^\infty : \sigma \in \Sigma_q^*\} = \bigcup_{I \in \Sigma_q^n} I\Gamma_q \quad (5.1)$$

for all $n \geq 1$. And recall that

$$\Lambda(\tau) = \{\Pi_b^\tau(J) : J \in \Gamma_q, \tau \text{ is regular on } J\},$$

where $\Pi_b^\tau(J) = \sum_{k=1}^\infty \tau(J|_k) b^{k-1}$. Denote naturally $\Pi_b^\tau(I) = \sum_{k=1}^n \tau(I|_k) b^{k-1}$ if $I \in \Sigma_q^n$, and $\Pi_{b,I}^\tau(J) = \sum_{k=1}^\infty \tau(Ij_1 \cdots j_k) b^{k-1}$ for $J = j_1 j_2 \cdots \in \Gamma_q$ where IJ is regular for τ . Define also

$$\Lambda_I(\tau) = \{\Pi_{b,I}^\tau(J) : J \in \Gamma_q, \tau \text{ is regular on } J\}.$$

By (5.1) we have

$$\Lambda(\tau) = \bigcup_{I \in \Sigma_q^n} (\Pi_b^\tau(I) + b^n \Lambda_I(\tau)).$$

The following is a simple lemma which was also observed in [DHS].

Proposition 5.2. *Let τ be a tree mapping and $n \geq 1$. Then $r\Lambda(\tau)$ is a spectrum for μ if and only if all $r\Lambda_I(\tau)$, $I \in \Sigma_q^n$, are spectra.*

Proof. Recall that μ_k satisfies $\widehat{\mu}(\xi) = \widehat{\mu}_k(\xi)\widehat{\mu}(b^{-k}\xi)$ and $\widehat{\mu}_k(\xi) = \prod_{j=1}^k m(b^{-j}\xi)$ where $m(\xi) = \frac{1}{q} \sum_{j=1}^q e^{2\pi i(j-1)\xi}$. Write $Q_I(\xi) = \sum_{\lambda \in \Lambda_I(\tau)} |\widehat{\mu}(\xi + r\lambda)|^2$. Note that

$$\begin{aligned} Q(\xi) &= \sum_{\lambda \in \Lambda(\tau)} |\widehat{\mu}(\xi + r\lambda)|^2 = \sum_{I \in \Sigma_q^n, \lambda \in \Lambda_I(\tau)} |\widehat{\mu}_n(\xi + r\Pi_b^\tau(I) + rb^n\lambda)|^2 \left| \widehat{\mu} \left(\frac{\xi + r\Pi_b^\tau(I)}{b^n} + r\lambda \right) \right|^2 \\ &= \sum_{I \in \Sigma_q^n, \lambda \in \Lambda_I(\tau)} |\widehat{\mu}_n(\xi + r\Pi_b^\tau(I))|^2 \left| \widehat{\mu} \left(\frac{\xi + r\Pi_b^\tau(I)}{b^n} + r\lambda \right) \right|^2 \\ &= \sum_{I \in \Sigma_q^n} |\widehat{\mu}_n(\xi + r\Pi_b^\tau(I))|^2 \cdot Q_I \left(\frac{\xi + r\Pi_b^\tau(I)}{b^n} \right). \end{aligned}$$

In a similar proof of Lemma 4.2, we have

$$1 \equiv \sum_{I \in \Sigma_q^n} |\widehat{\mu}_n(\xi + r\Pi_b^\tau(I))|^2.$$

Invoking Lemma 4.1, the result follows. \square

Proposition 5.2 asserted that spectral property are determined by a finite number of nodes. The following two lemmas show that the spectral property of a particular node σ can be determined by infinitely many of its offsprings and is *independent* of the regularity of $\sigma 0^\infty$. These are the key lemmas to the proof of Theorem 5.1.

Lemma 5.3. *Let $I \in \Sigma_q^*$ with $I \neq \varnothing$, the empty word. If τ is regular on $I0^\infty$, then*

$$\Lambda_I(\tau) = \{\Pi_{b,I}^\tau(0^\infty)\} \cup \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{q-1} (\Pi_{b,I}^\tau(0^{k-1}j) + b^k \Lambda_{I0^{k-1}j}(\tau)),$$

where $\Pi_{b,I}^\tau(0^{k-1}j) = \tau(I0) + \tau(I0^2)b + \cdots + \tau(I0^{k-1}j)b^{k-1}$. If τ is irregular on $I0^\infty$, then

$$\Lambda_I(\tau) = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{q-1} (\Pi_{b,I}^\tau(0^{k-1}j) + b^k \Lambda_{I0^{k-1}j}(\tau)).$$

Proof. Check it directly. \square

Lemma 5.4. *Let τ be a maximal mapping and let $I \in \Sigma_q^*$. Then $\Lambda_I(\tau)$ is a spectrum of μ if and only if $\Lambda_{I0^{k-1}j}(\tau)$ are spectra of μ for all $k \geq 1$ and $j = 1, \dots, q-1$.*

Proof. The necessity is clear from Proposition 5.2. We now prove the sufficiency. Assume that $\Lambda_{I0^{k-1}j}(\tau)$ are spectra for all $k \geq 1$ and $j = 1, \dots, q-1$. We need to show that $Q_I(\xi) = \sum_{\lambda \in \Lambda_I(\tau)} |\widehat{\mu}(\xi + r\lambda)|^2 \equiv 1$.

By the integral periodicity of m and Lemma 4.1 which will be used in the second equality below, we have for all $k \geq 2$,

$$\begin{aligned}
& \sum_{j=1}^{q-1} |\widehat{\mu}_k(\xi + r\Pi_{b,I}^\tau(0^{k-1}j))|^2 \\
&= \sum_{j=1}^{q-1} |\widehat{\mu}_{k-1}(\xi + r\Pi_{b,I}^\tau(0^{k-1}j))|^2 \left| \widehat{\mu}_1 \left(\frac{\xi + r\Pi_{b,I}^\tau(0^{k-1}j) + r\tau(0^{k-1}j)b^{k-1}}{b^k} \right) \right|^2 \\
&= |\widehat{\mu}_{k-1}(\xi + r\Pi_{b,I}^\tau(0^{k-1}j))|^2 \left(1 - \left| \widehat{\mu}_1 \left(\frac{\xi + r\Pi_{b,I}^\tau(0^k)}{b^k} \right) \right|^2 \right) \\
&= |\widehat{\mu}_{k-1}(\xi + r\Pi_{b,I}^\tau(0^{k-1}j))|^2 - |\widehat{\mu}_k(\xi + r\Pi_{b,I}^\tau(0^k))|^2.
\end{aligned}$$

If $k = 1$, the above becomes $\sum_{j=1}^{q-1} |\widehat{\mu}_1(\xi + r\Pi_{b,I}^\tau(j))|^2 = 1 - |\widehat{\mu}_1(\xi + r\Pi_{b,I}^\tau(0))|^2$. Now we simplify the following terms which is corresponding to the unions of the sets in Lemma 5.3,

$$\begin{aligned}
& \sum_{k=1}^{\infty} \sum_{j=1}^{q-1} \sum_{\lambda \in \Lambda_{\sigma 0^{k-1}j}(\tau)} |\widehat{\mu}(\xi + r\Pi_{b,I}^\tau(0^{k-1}j) + r b^k \lambda)|^2 \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{q-1} \sum_{\lambda \in \Lambda_{\sigma 0^{k-1}j}(\tau)} |\widehat{\mu}_k(\xi + r\Pi_{b,I}^\tau(0^{k-1}j))|^2 \left| \widehat{\mu} \left(\frac{\xi + r\Pi_{b,I}^\tau(0^{k-1}j)}{b^k} + \lambda \right) \right|^2 \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{q-1} |\widehat{\mu}_k(\xi + r\Pi_{b,I}^\tau(0^{k-1}j))|^2 \\
&= 1 - \lim_{N \rightarrow \infty} |\widehat{\mu}_N(\xi + r\Pi_{b,I}^\tau(0^N))|^2 \\
&= 1 - \prod_{j=1}^{\infty} \left| m \left(\frac{\xi + r\Pi_{b,I}^\tau(0^j)}{b^j} \right) \right|^2. \tag{5.2}
\end{aligned}$$

We now divide the proof into two cases.

Case (i). If $I0^\infty$ is regular, then $|\widehat{\mu}(\xi + r\Pi_{b,I}^\tau(0^\infty))|^2 = \prod_{j=1}^{\infty} \left| m \left(\frac{\xi + r\Pi_{b,I}^\tau(0^j)}{b^j} \right) \right|^2$. Hence, by Lemma 5.3,

$$Q_I(\xi) = |\widehat{\mu}(\xi + r\Pi_{b,I}^\tau(0^\infty))|^2 + (5.2) \equiv 1.$$

This shows $r\Lambda_I(\tau)$ is a spectrum.

Case (ii). If $I0^\infty$ is irregular, then Lemma 5.3 shows that

$$Q_I(\xi) = (5.2) = 1 - \prod_{j=1}^{\infty} \left| m \left(\frac{\xi + r\Pi_{b,I}^\tau(0^j)}{b^j} \right) \right|^2.$$

Note that, by (ii) in the definition of the maximal mapping, we may write

$$r\Pi_{b,I}(0^n) = r \sum_{j=1}^n \tau(I0^j) b^{j-1} = r \sum_{j=1}^n (s_j q) b^{j-1} = \sum_{j=1}^n s_j b^j \quad (5.3)$$

for $s_j \in \{0, 1, \dots, r-1\}$. Then

$$Q_I(\xi) = 1 - \prod_{j=1}^{\infty} \left| m \left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^{j-1})}{b^j} \right) \right|^2. \quad (5.4)$$

Suppose on the contrary $Q_I(\xi) < 1$ for some $\xi > 0$. Since Q_I is entire, we may assume ξ is small, say $|\xi| < \frac{r-1}{b-1}$. From (5.4), we must have $\prod_{j=1}^{\infty} \left| m \left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^{j-1})}{b^j} \right) \right|^2 > 0$.

For those n such that $s_n \neq 0$ in (5.3).

$$\frac{1}{b} \leq \frac{s_n}{b} \leq \left| \frac{\xi + \sum_{j=1}^n s_j b^j}{b^{n+1}} \right| \leq \frac{r-1}{b(b-1)} < \frac{1}{q(b-1)}.$$

Hence, letting $c = \max\{|m(\xi)|^2 : \frac{1}{b} \leq |\xi| < \frac{1}{q(b-1)}\} < 1$, we have

$$\left| m \left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^n)}{b^{n+1}} \right) \right|^2 = \left| m \left(\frac{\xi + \sum_{j=1}^n s_j b^j}{b^{n+1}} \right) \right|^2 \leq c.$$

And

$$\prod_{j=1}^{\infty} \left| m \left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^{j-1})}{b^j} \right) \right|^2 = \lim_{N \rightarrow \infty} \prod_{j=1}^N \left| m \left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^{j-1})}{b^j} \right) \right|^2 \leq \lim_{N \rightarrow \infty} c^{\#\{n: s_n \neq 0, n \leq N\}}.$$

As $I0^\infty$ is irregular, there exists infinitely many $s_j \neq 0$. The above limit is zero. This is a contradiction and hence $\Lambda_I(\tau)$ must be a spectrum. \square

Proof of Theorem 5.1. By the definition of τ and τ' , $\Lambda_{I'}(\tau) = \Lambda_{I'}(\tau')$ for all $I' \neq I$ and $I' \in \Sigma_q^k$. Moreover, $\Lambda_{I0^{n-1}j}(\tau) = \Lambda_{I0^{k-1}j}(\tau')$ for all $k \geq 1$ and $j = 1, \dots, q-1$. Therefore, if τ is a spectrum, then $\Lambda_{I'}(\tau')$ are spectra of μ for all $\sigma' \neq \sigma$ and $\sigma' \in \Sigma_q^k$ by Proposition 5.2. On the other hand, $\Lambda_{I0^{k-1}j}(\tau')$ are spectra of μ also as τ is a spectrum. By Lemma 5.4, $\Lambda_I(\tau')$ is also a spectrum. We therefore conclude that $\Lambda(\tau')$ is a spectrum of μ by Proposition 5.2 again. The converse also holds by reversing the role of τ and τ' . This completes the whole proof. \square

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